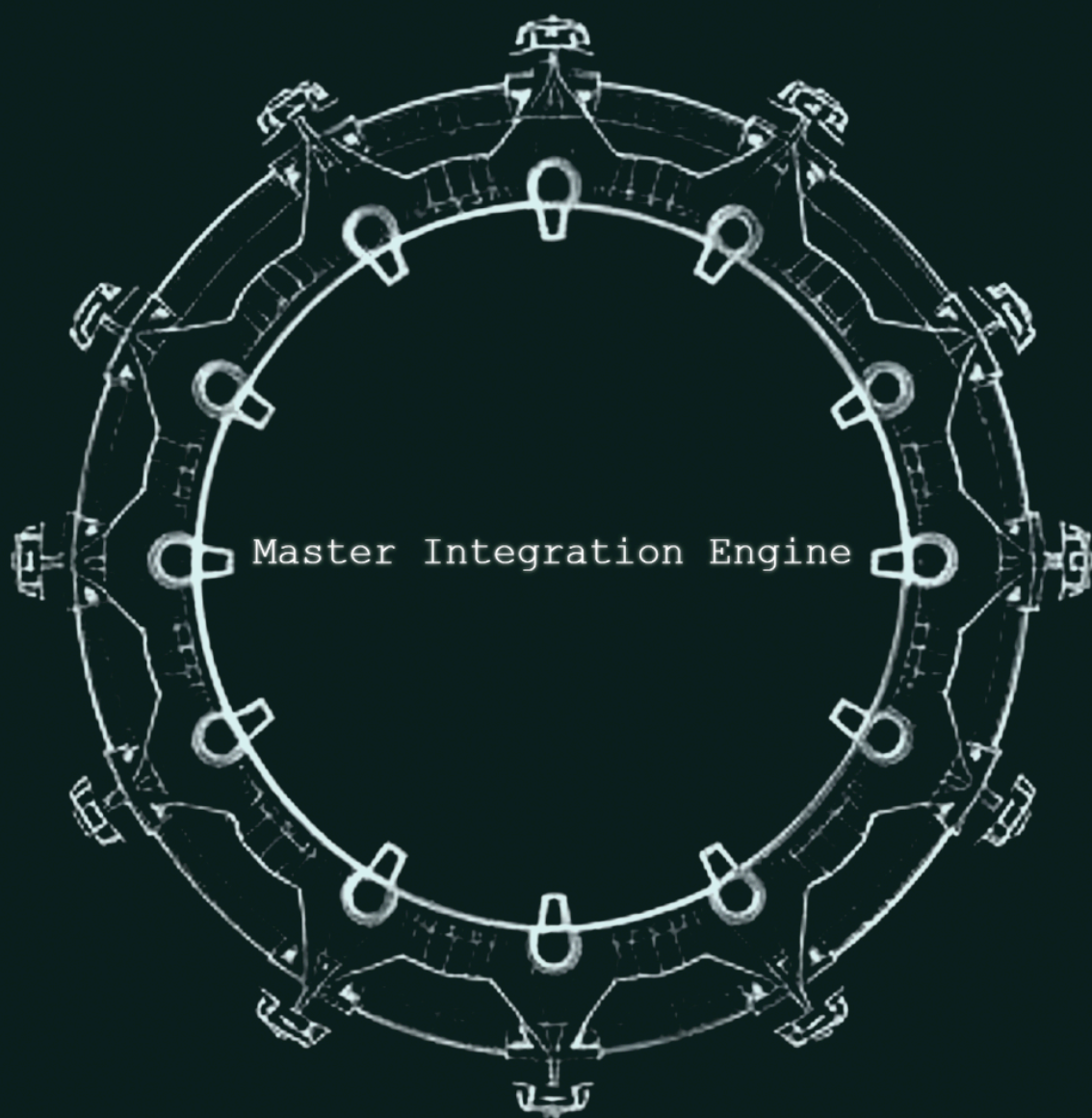




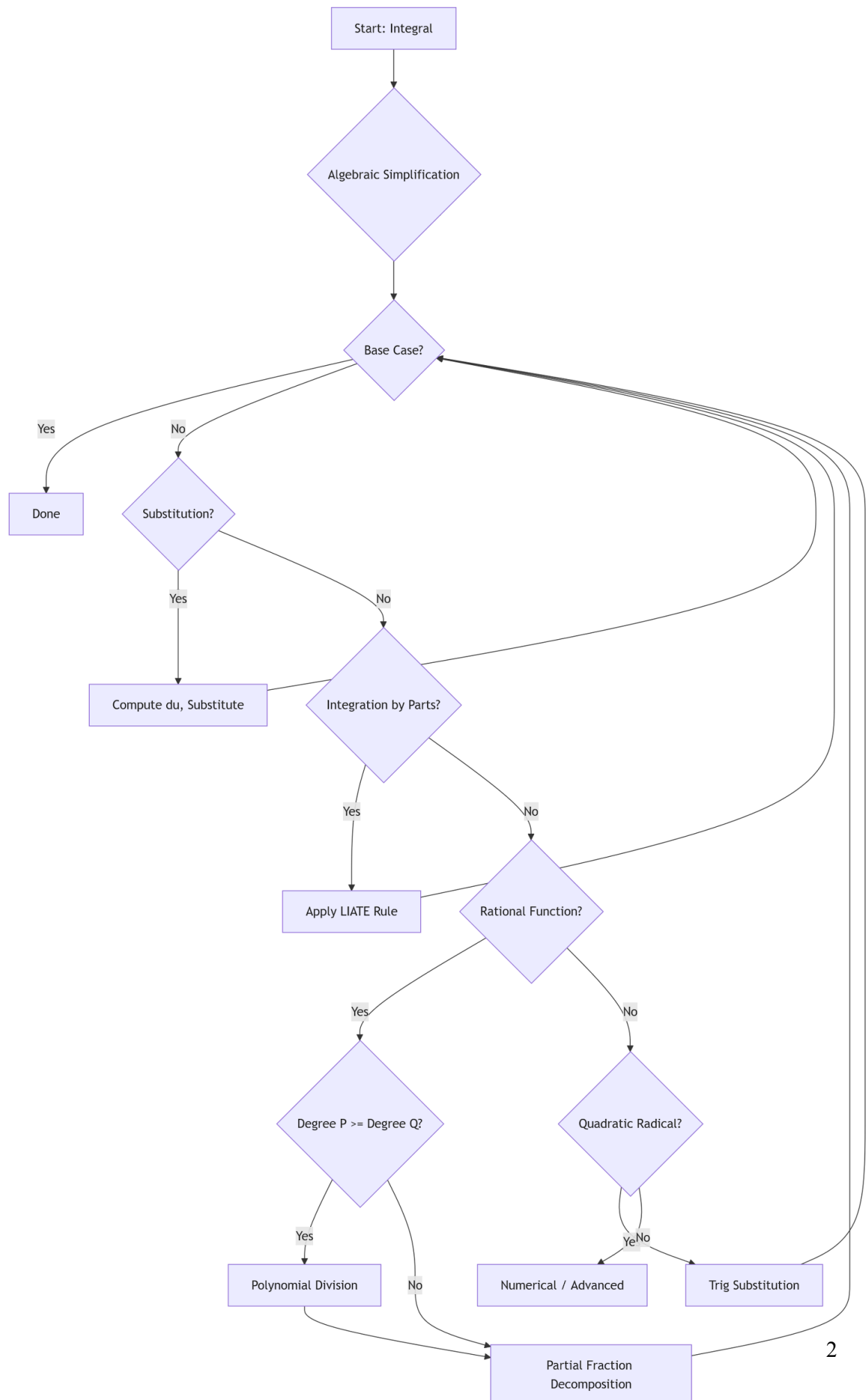
Master Integration Engine



I'm going to level with you—the world of integration can feel like an absolute labyrinth sometimes. I've stared at a blank page more times than I care to admit, wondering why a simple-looking integral has decided to be so stubborn. But here is the thing: it's not you, and it's certainly not a failure of your intelligence. It's just that integration is a puzzle, not a mechanical process.

I've put together this framework not to make you feel like you have to be a machine, but to give you a map so you don't have to wander alone:

Acrilic's Master Integration Engine.....	3
1. Phase Zero: The "Clean Up".....	3
2. The Destination: Base Cases.....	3
3. Choosing Your Path (The Decision Engine).....	4
1. The Riemann Integral and Fundamental Theorems.....	5
1.1. Definite Integral (Riemann Concept).....	5
1.2. Fundamental Theorem of Calculus (FTC).....	5
2. Calculation of Primitives.....	6
2.1. Table of Fundamental Immediate Primitives.....	6
2.2. Change of Variable.....	7
2.3. Integration by Parts.....	7
3. Improper Integrals.....	7
3.1. Species I: Infinite Intervals.....	7
3.2. Species II: Infinite Discontinuities.....	8
3.3. Limit Comparison Criterion.....	8
4. Introduction to Ordinary Differential Equations (ODE).....	8
4.1. Fundamental Concepts.....	8
4.2. Separable Variable ODEs.....	9
4.3. Classic Application: Newton's Law of Cooling.....	9
4.4. First-Order Linear ODEs.....	9
4.5. Physical Modeling and Dynamic Phenomena.....	10
5. Geometric Applications of the Integral.....	10
5.1. Calculation of the Area under a Curve.....	10
5.2. Area between two curves.....	11



Acrilic's Master Integration Engine

Integration is an iterative process of pattern recognition. This framework is designed to help you systematically break down complex integrands into pieces you actually recognize.

1. Phase Zero: The "Clean Up"

Before you get into the heavy math, try to make the problem look friendlier. Skipping this is usually where the headache starts.

- **Expand Polynomials:** Don't hesitate to multiply things out; it often reveals that a complex problem is just several simple ones hiding in disguise.
- **Split Fractions:** If you have something like $\frac{A+B}{C}$, rewrite it as $\frac{A}{C} + \frac{B}{C}$. It is much easier to tackle two small, manageable integrals than one giant one.
- **Trigonometric Reduction:** Use power-reduction identities (like $\sin^2(x) = \frac{1-\cos(2x)}{2}$) to turn high-order powers into something linear. It's a bit of a trick, but it saves so much time.

2. The Destination: Base Cases

Everything we do is just trying to get the math to look like these building blocks. If you reach these, you've done the hard part.

Function $f(x)$	Integral $\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} + C \ (n \neq -1)$
$1/x$	$\ln x + C$
e^{ax}	$\frac{1}{a}e^{ax} + C$
$\frac{1}{1+x^2}$	$\arctan(x) + C$

3. Choosing Your Path (The Decision Engine)

- **Substitution (u -substitution):** Think of this as a “variable transplant”. If you see a function and its derivative sitting right next to it, define u as the inner function and watch the complexity collapse. If you end up with leftover x terms you can't get rid of, don't worry—just hit the reset button and try another path.
- **Integration by Parts:** This is for when you have two different types of functions multiplied together (like $x \cdot \sin(x)$). Use the **ALPES** order (Arc-trig, Log, Polynomial, Exponential, Sine/Cosine) to pick your u . The goal is to pick a u that gets *simpler* when you differentiate it.
- **Partial Fraction Decomposition:** If you're staring at a big, messy fraction, divide the top by the bottom first if you need to. Then, break the bottom into its factors. It's a bit of algebra, but it's a very predictable, safe process.
- **Trig Substitution:** If you see a square root with x^2 inside, it's a hint to use a trig identity to “collapse” the radical. It feels like magic, but it's just geometry.

As per how to tackle integration methods and understand integration, here is your guide so you don't go insane.

1. The Riemann Integral and Fundamental Theorems

1.1. Definite Integral (Riemann Concept)

This is the fundamental concept of the entire syllabus. The definite integral generalizes the calculation of the area under a curve through infinitesimal partitions of the domain.

If $f(x)$ is a continuous function (or one with a finite number of jump discontinuities) in the closed interval $[a, b]$, the definite integral is denoted and conceptually defined by the limit of Riemann sums:

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x$$

Properties of the Definite Integral

- **Linearity:** $\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$
- **Interval Additivity:** $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad \forall c \in (a, b)$
- **Inversion of Limits:** $\int_a^b f(x)dx = - \int_b^a f(x)dx$

1.2. Fundamental Theorem of Calculus (FTC)

Aside from the aforementioned, the FTC—or as I like to call it: the *overt theorem*—establishes the bidirectional connection between differential calculus and integral calculus.

First Part:

If f is continuous on $[a, b]$, the integral function $F(x)$ defined by $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b) , satisfying:

$$F'(x) = f(x)$$

Differentiation of composite integral functions

By the Chain Rule, if the limits of integration are functions $u(x)$ and $v(x)$, the derivative is:

$$\frac{d}{dx} \left(\int_{v(x)}^{u(x)} f(t) dt \right) = f(u(x)) \cdot u'(x) - f(v(x)) \cdot v'(x)$$

Second Part or Barrow's Rule:

If $G(x)$ is any **primitive** of $f(x)$ (that is, $G'(x) = f(x)$), then:

$$\int_a^b f(x) dx = G(b) - G(a) = [G(x)]_a^b$$

This pretty much explains the relationship between the integral and the derivative, where one will feel like a wall in the park and the other like you want to gauge your eyes out.

2. Calculation of Primitives

2.1. Table of Fundamental Immediate Primitives

General Rule for Polynomials

Due to the linearity property, the integral of a polynomial is solved by integrating term-by-term using the power rule. The multiplying constants are kept, and independent terms add an x .

For a generic polynomial of degree n :

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Its exact indefinite integral is:

$$\int P(x) dx = a_n \frac{x^{n+1}}{n+1} + a_{n-1} \frac{x^n}{n} + \cdots + a_1 \frac{x^2}{2} + a_0 x + C$$

Function $f(x)$	Primitive $\int f(x)dx$	Condition
x^n	$\frac{x^{n+1}}{n+1} + C$	$n \neq -1$
$\frac{1}{x}$	$\ln x + C$	$x \neq 0$
e^{ax}	$\frac{1}{a}e^{ax} + C$	$a \neq 0$
$\sin(x)$	$-\cos(x) + C$	
$\cos(x)$	$\sin(x) + C$	
$\frac{1}{1+x^2}$	$\arctan(x) + C$	
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x) + C$	

The Constant of Integration

Never forget to add the scalar constant $+C$ when solving indefinite integrals—else probably your professors will kill you during your exam. Omitting C implies the loss of infinite valid solutions that differ by a vertical translation.

2.2. Change of Variable

Based on the chain rule. It is used when the integrand is (or resembles) the product of a composite function and the derivative of its argument.

Let $\int f(g(x))g'(x)dx$. Define $u = g(x)$, which implies $du = g'(x)dx$.

$$\int f(g(x))g'(x)dx = \int f(u)du$$

2.3. Integration by Parts

Derived from the product rule for derivatives: $(uv)' = u'v + uv'$. Integrating both sides yields the fundamental formula:

$$\int u dv = u \cdot v - \int v du$$

ALPES Mnemonic Rule to choose u

Now it is really important to follow this, unless you want to be stuck in an infinite loop. To choose the function u (the one to be differentiated), pick the first one that appears in this order: **A**rcos (inverse trig), **L**ogarithmic, **P**olynomial, **E**xponential, **S**ine/cosine (Trig). The remainder of the integrand will be dv .

3. Improper Integrals

Improper integrals extend the Riemann integral concept to non-bounded intervals or integrands with asymptotic discontinuities. This is the basis of one the greatest properties of integrals aside from wanting to jump off a bridge.

3.1. Species I: Infinite Intervals

They are evaluated analytically using limits:

$$\int_a^\infty f(x)dx = \lim_{b \rightarrow \infty} \int_a^b f(x)dx$$

- If the limit exists and is finite, the integral **converges**.
- If the limit is infinite or does not exist, the integral **diverges**.

p -Integrals (Standard Reference Criterion)

$$\int_1^{\infty} \frac{1}{x^p} dx$$

- **Converges** if $p > 1$.
- **Diverges** if $p \leq 1$.

3.2. Species II: Infinite Discontinuities

If $f(x)$ presents a vertical asymptote at $x = b$ (i.e., $\lim_{x \rightarrow b^-} f(x) = \pm\infty$):

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx$$

3.3. Limit Comparison Criterion

Let $f(x) \geq 0$ and $g(x) \geq 0$. If the asymptotic limit is calculated:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$$

If $0 < L < \infty$, then $\int_a^{\infty} f(x) dx$ and $\int_a^{\infty} g(x) dx$ share the same character (both converge or both diverge).

4. Introduction to Ordinary Differential Equations (ODE)

4.1. Fundamental Concepts

An **Ordinary Differential Equation (ODE)** relates an independent variable x , an unknown function $y(x)$, and its derivatives $y', y'', \dots, y^{(n)}$. The **order** of the ODE is determined by the highest-degree derivative appearing in the equation.

Initial Value Problem (IVP)

The general solution of a first-order ODE includes a constant C . To find a unique particular solution, a mathematical initial condition of the type $y(x_0) = y_0$ is required. The combination of the ODE and the initial condition is called an IVP.

4.2. Separable Variable ODEs

These are first-order equations that can be factored algebraically so that all terms involving y are on one side, and those involving x are on the other.

Acrylic

Canonical structure:

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

Resolution Algorithm:

1. Separate the differentials by operating algebraically:

$$\frac{1}{g(y)} dy = f(x) dx$$

2. Integrate both sides simultaneously:

$$\int \frac{1}{g(y)} dy = \int f(x) dx + C$$

3. Solve for $y(x)$ (if possible) to obtain the explicit solution.

Now you'll most likely encounter one of the most common applications of ODE's, again from our favourite virgin: Newton.

4.3. Classic Application: Newton's Law of Cooling

The rate of change of the temperature $T(t)$ of a body is proportional to the difference between its temperature and the ambient temperature T_a :

$$\frac{dT}{dt} = k(T - T_a)$$

Since it has separable variables, its resolution implies:

$$\int \frac{dT}{T - T_a} = \int k dt \implies \ln ||T - T_a|| = kt + C$$

$$T(t) = T_a + A \cdot e^{kt} \quad \text{where } A = \pm e^C$$

The constant k (typically negative for cooling) and the geometric constant A are determined by substituting the initial conditions of the actual physical problem.

4.4. First-Order Linear ODEs

A first-order linear differential equation has the standard form:

$$y' + p(x)y = q(x)$$

- If $q(x) = 0$, the equation is **homogeneous** and can be solved by the method of separable variables.
- If $q(x) \neq 0$, the equation is **non-homogeneous**.

Resolution Method:

To solve the complete ODE, the entire equation is multiplied by an auxiliary function called the **integrating factor** $\mu(x)$, defined as:

$$\mu(x) = e^{\int p(x)dx}$$

This factor transforms the left side of the equation into the exact derivative of a product, allowing for direct integration:

$$(\mu(x) \cdot y)' = \mu(x) \cdot q(x)$$

$$y(x) = \frac{1}{\mu(x)} \left(\int \mu(x)q(x)dx + C \right)$$

4.5. Physical Modeling and Dynamic Phenomena

ODEs are the primary engineering tool for modeling rates of change. Based on the problems proposed in the syllabus, the following structural patterns stand out:

Classic 1st Order Models

1. **Free fall with air resistance:** Acceleration (rate of change of velocity) depends on gravity and friction proportional to velocity.

$$\frac{dv}{dt} = g - k \cdot v$$

2. **Learning / Memorization Curve:** The speed of memorization is proportional to the amount of material *remaining* to be studied. If the goal is M :

$$\frac{dy}{dt} = k(M - y)$$

3. **Proportional Growth/Decay:** Populations, radioactive decay, or continuous interest.

$$\frac{dy}{dt} = k \cdot y \implies y(t) = y_0 e^{kt}$$

5. Geometric Applications of the Integral

5.1. Calculation of the Area under a Curve

The definite Riemann integral $\int_a^b f(x)dx$ represents the net oriented area. To calculate the **actual geometric area** (which must always be strictly positive), the absolute value of the function must be integrated:

$$Area = \int_a^b |f(x)|dx$$

Calculation Algorithm:

1. Find the intersection points of $f(x)$ with the X-axis by solving $f(x) = 0$.
2. Divide the original integration interval $[a, b]$ using the roots found that fall within it.
3. Integrate each subinterval separately and sum the absolute values of the results.

5.2. Area between two curves

To calculate the area enclosed between two functions $f(x)$ and $g(x)$ in the interval $[a, b]$:

$$Area = \int_a^b |f(x) - g(x)| dx$$

Determination of Integration Limits

In engineering exams, the limits a and b are rarely provided explicitly. It is assumed that the area is naturally limited by the intersection points of the two curves.

Critical Step: Equate the functions $f(x) = g(x)$ and solve the equation algebraically to obtain the integration limits $x_1, x_2, \dots, x_n$.

Roof Function vs. Floor Function Heuristic

To avoid using absolute values within the integral, evaluate an intermediate point in the intersection interval to determine which function is "above" (Roof) and which is "below" (Floor).

The area without absolute value is directly:

$$Area = \int_a^b [f_{\text{roof}}(x) - g_{\text{floor}}(x)] dx$$

In conclusion, the transition from viewing integration as a series of disparate, headache-inducing tasks to seeing it as a structured “Master Engine” is the ultimate milestone of this paper.

You’ve moved past the initial phase of blindly attempting substitutions, trading that uncertainty for a systematic diagnostic workflow—the “Clean Up”, the identification of “Base Cases”, and the strategic application of the “Decision Engine”. Whether you were wrestling with the absolute value requirements of area calculations, or navigating the boundary conditions of improper integrals, the core realization remains: **Integration is fundamentally a game of algebraic management.**

By treating functions as entities that require deliberate transformation before integration—rather than objects to be attacked with the first tool at hand—you have developed the technical intuition required for higher-level engineering. The rigor you’ve applied to defining the “Roof vs. Floor” heuristic and separating variables in ODEs isn’t just about passing an exam; it’s about establishing the deterministic control needed for physical modeling and dynamic analysis.